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MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Acoustic Shadow of a Sphere. With an Appendix, Giving the Values of Legendre's Functions from P $_{0}$ to P$_{20}$ at Intervals of 5 Degrees

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Phil. Trans. R. Soc. Lond. A 1904 **203**, 87-110
doi: 10.1098/rsta.1904.0016

IV. *On the Acoustic Shadow of a Sphere.*

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With an Appendix, giving the Values of LEGENDRE'S *Functions from* P_0 *to* P_{20} *at Intervals of* 5 *degrees. By Professor* A. LODGE.

Received December 28, 1903,—Read January 21, 1904.

IN my book on the 'Theory of Sound,' § 328, I have discussed the effect upon a source of sound of a rigid sphere whose surface is close to the source.

The question turns upon the relative magnitudes of the wave-length (λ) and the radius (c) of the sphere. If kc be *small*, where $k = 2\pi/\lambda$, the presence of the sphere has but little effect upon the sound to be perceived at a distance.

The following table was given, showing the effect in three principal directions of somewhat larger spheres :—

kc.	μ.	$F^2 + G^2$.
$\frac{1}{2}$	1 −1 0	·294291 ·259729 ·231999
1	1 −1 0	·502961 ·285220 ·236828
2	1 −1 0	·6898 ·3182 ·3562

Here $F^2 + G^2$ represents the intensity of sound at a great distance from the sphere in directions such that μ is the cosine of the angle between them and that radius which passes through the source. Upon the scale of measurement adopted, $F^2 + G^2 = \frac{1}{4}$ for all values of μ, when $kc = 0$, that is, when the propagation is undisturbed by any obstacle. The increased values under $\mu = 1$ show that the sphere is beginning to act as a *reflector*, the intensity in this direction being already more than doubled when $kc = 2$. "In looking at these figures, the first point which attracts attention is the comparatively slight deviation from uniformity in the intensities in different directions. Even when the circumference of the sphere amounts to twice the wave-length, there is scarcely anything to be called a sound shadow. But what is, perhaps, still more unexpected is that in the first two cases the intensity behind the sphere [$\mu = -1$] exceeds that in a transverse direction [$\mu = 0$]. This result depends mainly on the preponderance of the term of the first order, which vanishes with μ. The order of the more important terms increases with kc; when kc is 2, the principal term is of the second order.

"Up to a certain point the augmentation of the sphere will increase the total energy emitted, because a simple source emits twice as much energy when close to a rigid plane as when entirely in the open. Within the limits of the table this effect masks the obstruction due to an increasing sphere, so that when $\mu = -1$, the intensity is greater when the circumference is twice the wave-length than when it is half the wave-length, the source itself remaining constant."

The solution of the problem when kc is very great cannot be obtained by this method, but it is to be expected that when $\mu = 1$ the intensity will be quadrupled, as when the sphere becomes a plane, and that when μ is negative the intensity will tend to vanish. It is of interest to trace somewhat more closely the approach to this state of things—to treat, for example, the case of $kc = 10$.* In every case where it can be carried out the solution has a double interest, since in virtue of the law of reciprocity it applies when the source and point of observation are interchanged, thus giving the intensity at a point on the sphere due to a source situated at a great distance.

But before proceeding to consider a higher value of kc, it will be well to supplement the information already given under the head of $kc = 2$. The original calculation was limited to the principal values of μ, corresponding to the poles and the equator, under the impression that results for other values of μ would show nothing distinctive. The first suggestion to the contrary was from experiment. In observing the shadow of a sphere, by listening through a tube whose open end was presented to the sphere, it was found that the somewhat distant source was *more* loudly heard at the anti-pole ($\mu = -1$) than at points 40° or 50° therefrom. This is analogous to POISSON's experiment, where a bright point is seen in the centre of the shadow of a circular disc—an experiment easily imitated acoustically†—and it may be generally explained in the same manner. This led to further calculations for values of μ between 0 and -1, giving numbers in harmony with observation. The complete results for this case ($kc = 2$) are recorded in the annexed table. In obtaining them, terms of LEGENDRE's series, up to and including P_6, were retained. The angles θ are those whose cosine is μ.

* See RAYLEIGH, 'Proc. Roy. Soc.,' vol. 72, p. 40; also MACDONALD, vol. 71, p. 251; vol. 72, p. 59; POINCARÉ, vol. 72, p. 42.

† 'Phil. Mag.,' vol. 9, p. 278, 1880; 'Scientific Papers,' vol. 1, p. 472.

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$kc = 2.$

θ.	F + *i*G.	F² + G².	4 (F² + G²).
° 0	+ ·7968 + ·2342 *i*	·6898	2·759
15	+ ·8021 + ·1775 *i*	·6749	2·700
30	+ ·7922 + ·0147 *i*	·6278	2·511
45	+ ·7139 − ·2287 *i*	·5619	2·248
60	+ ·5114 − ·4793 *i*	·4912	1·965
75	+ ·1898 − ·6247 *i*	·4263	1·705
90	− ·1538 − ·5766 *i*	·3562	1·425
105	− ·3790 − ·3413 *i*	·2601	1·040
120	− ·3992 − ·0243 *i*	·1600	0·640
135	− ·2401 + ·2489 *i*	·1196	0·478
150	− ·0088 + ·4157 *i*	·1729	0·692
165	+ ·1781 + ·4883 *i*	·2701	1·080
180	+ ·2495 + ·5059 *i*	·3182	1·273

A plot of $4\,(F^2 + G^2)$ against θ is given in fig. 1, curve A.

The investigation for $kc = 10$ could probably be undertaken with success upon the lines explained in 'Theory of Sound;' but as it is necessary to include some 20 terms

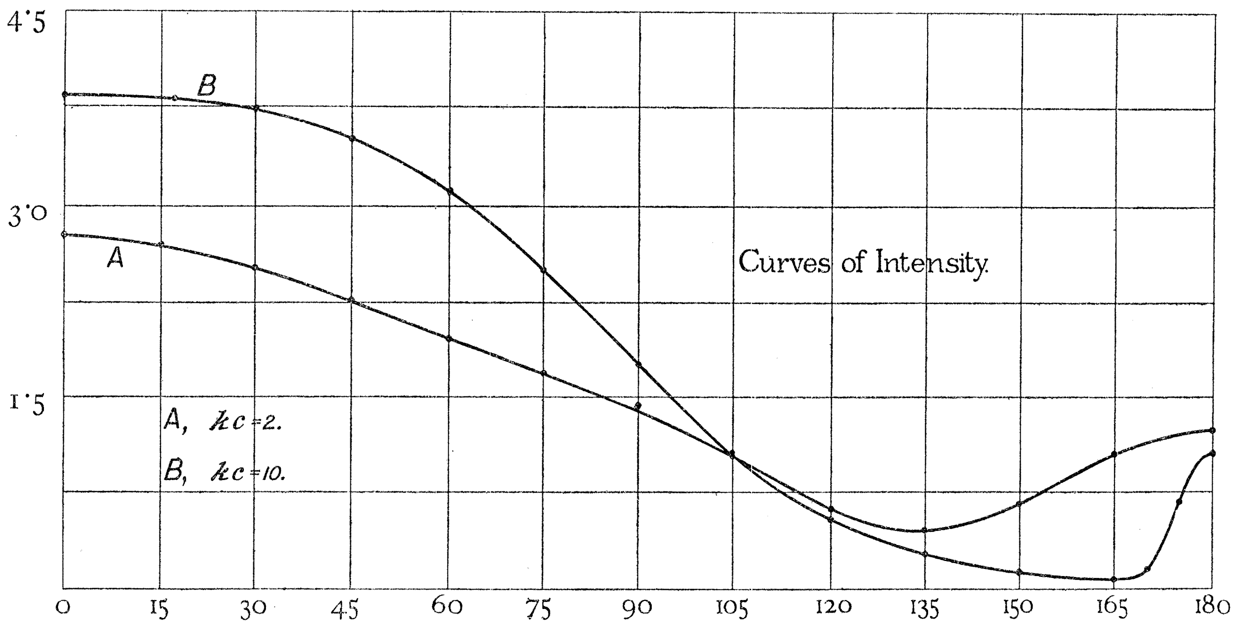

Fig. 1.

of the expansion in LEGENDRE's series, I considered that it would be advantageous to use certain formulæ of reduction by which the functions of various orders can be deduced from their predecessors, and this involves a change of notation. Formulæ convenient for the purpose have been set out by Professor LAMB.* The velocity-

* 'Hydrodynamics,' § 267; 'Camb. Phil. Trans.,' vol. 18, p. 350, 1900.

potential ψ is supposed to be proportional throughout to e^{ikat}, but this time-factor is usually omitted. The general differential equation satisfied by ψ is

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + \frac{d^2\psi}{dz^2} + k^2\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad (1),$$

of which the solution in polar co-ordinates applicable to a *divergent* wave of the n^{th} order in LAPLACE's series may be written

$$\psi_n = \mathrm{S}_n r^n \chi_n(kr) \quad . \quad . \quad . \quad . \quad . \quad (2).$$

For the present purpose we may suppose without loss of generality that $k = 1$. The differential equation satisfied by $\chi_n(r)$ is

$$\frac{d^2\chi_n}{dr^2} + \frac{2n+2}{r}\frac{d\chi_n}{dr} + \chi_n = 0 \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and of this the solution which corresponds to a divergent wave is

$$\chi_n(r) = \left(-\frac{d}{r\,dr}\right)^n \frac{e^{-ir}}{r} \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Putting $n = 0$ and $n = 1$, we have

$$\chi_0(r) = \frac{e^{-ir}}{r}, \qquad \chi_1(r) = \frac{(1+ir)\,e^{-ir}}{r^3} \quad . \quad . \quad . \quad . \quad (5).$$

It is easy to verify that (4) satisfies (3). For if χ_n satisfies (3), $r^{-1}\chi'_n$ satisfies the corresponding equation for χ_{n+1}. And $r^{-1}e^{-ir}$ satisfies (3) when $n = 0$.

From (3) and (4) the following formulæ of reduction may be verified:

$$\chi'_n(r) = -r\chi_{n+1}(r) \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$r\chi'_n(r) + (2n+1)\chi_n(r) = \chi_{n-1}(r) \quad . \quad . \quad . \quad . \quad (7),$$

$$\chi_{n+1}(r) = \frac{(2n+1)\chi_n(r) - \chi_{n-1}(r)}{r^2} \quad . \quad . \quad . \quad . \quad (8).$$

By means of the last, χ_2, χ_3, &c., may be built up in succession from χ_0 and χ_1.

From (2)

$$d\psi_n/dr = \mathrm{S}_n(nr^{n-1}\chi_n + r^n\chi'_n),$$

or, with use of (7),

$$d\psi_n/dr = r^{n-1}\mathrm{S}_n\{\chi_{n-1} - (n+1)\chi_n\} \quad . \quad . \quad . \quad . \quad (9).$$

Thus, if U_n be the n^{th} component of the normal velocity at the surface of the sphere ($r = c$),

$$\mathrm{U}_n = c^{n-1}\mathrm{S}_n\{\chi_{n-1}(c) - (n+1)\chi_n(c)\} \quad . \quad . \quad . \quad . \quad (10).$$

When $n = 0$,

$$\mathrm{U}_0 = \mathrm{S}_0\chi'_0(c) = -\mathrm{S}_0 c\chi_1(c) \quad . \quad . \quad . \quad . \quad (11).$$

The introduction of S_n from (10), (11) into (2) gives ψ_n in terms of U_n supposed known.

When r is very great in comparison with the wave-length, we get from (4)

$$\chi_n(r) = \frac{i^n e^{-ir}}{r^{n+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

so that

$$\psi_n = S_n \frac{i^n e^{-ir}}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

In order to find the effect at a great distance of a source of sound localised on the surface of the sphere at the point $\mu = 1$, we have only to expand the complete value of U in LEGENDRE's functions. Thus

$$U_n = \tfrac{1}{2}(2n+1) P_n(\mu) \int_{-1}^{+1} U P_n(\mu)\, d\mu$$

$$= \tfrac{1}{2}(2n+1) P_n(\mu) \int_{-1}^{+1} U\, d\mu = \frac{2n+1}{4\pi c^2} P_n(\mu) \iint U\, dS \quad . \quad . \quad (14),$$

in which $\iint U\, dS$ denotes the magnitude of the *source*, *i.e.*, the integrated value of U over the small area where it is sensible. The complete value of ψ may now be written

$$\psi = \frac{\iint U\, dS\,.\, e^{i(at-r)}}{4\pi r} \sum \frac{(2n+1)\, i^n P_n(\mu)}{c^{n+1}\{\chi_{n-1}(c) - (n+1)\chi_n(c)\}} \quad . \quad . \quad . \quad (15).$$

When $n = 0$, $\chi_{n-1}(c) - (n+1)\chi_n(c)$ is to be replaced by $-c^2\chi_1(c)$.

If we compare (15) with the corresponding expression in "Theory of Sound," (3), § 238, we get

$$c^{n+1}\{\chi_{n-1}(c) - (n+1)\chi_n(c)\} = -i^n e^{-ic} F_n(ic) \quad . \quad . \quad . \quad . \quad (16).$$

Another particular case of interest arises when the point of observation, as well as the source, is on the sphere, so that, instead of $r = \infty$, we have $r = c$. Equation (15) is then replaced by

$$\psi = \frac{\iint U\, dS\,.\, e^{ikat}}{4\pi c} \sum \frac{(2n+1)\chi_n(c) P_n(\mu)}{\chi_{n-1}(c) - (n+1)\chi_n(c)} \quad . \quad . \quad . \quad . \quad . \quad (17).$$

It may be remarked that, since ψ in (17) is infinite when $\mu = +1$ and accordingly $P_n = 1$, the convergence at other points can only be attained in virtue of the factors P_n. The difficulties in the way of a practical calculation from (17) may be expected to be greater than in the case of (15).

We will now proceed to the actual calculation for the case of $c = 10$, or $kc = 10$. The first step is the formation of the values of the various functions $\chi_n(10)$, starting from $\chi_0(10)$, $\chi_1(10)$. For these we have from (5)

$$10\chi_0(10) = \cos 10 - i \sin 10,$$

$$10^2\chi_1(10) = \tfrac{1}{10}\cos 10 + \sin 10 + i(\cos 10 - \tfrac{1}{10}\sin 10).$$

The angle (10 radians) $= 540° + 32°\ 57'{\cdot}468$; thus

$$\sin 10 = -{\cdot}5440210, \quad \cos 10 = -{\cdot}8390716,$$

and

$$10\chi_0 = -{\cdot}8390716 + {\cdot}5440210\, i,$$

$$10^2\chi_1 = -{\cdot}6279282 - {\cdot}7846695\, i.$$

From these, χ_2, χ_3, . . . are to be computed in succession from (8), which may be put into the form

$$10^{n+2}\chi_{n+1} = \frac{2n+1}{10}\, 10^{n+1}\chi_n - 10^n\chi_{n-1}.$$

For example,

$$10^3\chi_2 = {\cdot}3\,(10^2\chi_1) - 10\chi_0 = +{\cdot}6506931 - {\cdot}7794218\, i.$$

When the various functions $10^{n+1}\chi_n$ have been computed, the next step is the computation of the denominators in (15). We write

$$\begin{aligned} D_n &= 10^{n+1}\{\chi_{n-1} - (n+1)\chi_n\} \\ &= 10 \times 10^n\chi_{n-1} - (n+1)\, 10^{n+1}\chi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (18), \end{aligned}$$

and the values of D_n are given along with $10^{n+1}\chi_n$ in the annexed table.

n.	$10^{n+1}\chi_n(10)$.	D_n.
0	$-$ 0·83907 + 0·54402 i	+ 6·2793 + 7·8467 i
1	$-$ 0·62793 − 0·78467 i	$-$ 7·1349 + 7·0095 i
2	+ 0·65069 − 0·77942 i	$-$ 8·2314 − 5·5084 i
3	+ 0·95327 + 0·39496 i	+ 2·6938 − 9·3741 i
4	+ 0·01660 + 1·05589 i	+ 9·4498 − 1·3299 i
5	$-$ 0·93834 + 0·55534 i	+ 5·7960 + 7·2269 i
6	$-$ 1·04877 − 0·44501 i	$-$ 2·0420 + 8·6685 i
7	$-$ 0·42506 − 1·13386 i	$-$ 7·0872 + 4·6208 i
8	+ 0·41117 − 1·25578 i	$-$ 7·9512 − 0·0366 i
9	+ 1·12406 − 1·00096 i	$-$ 7·1288 − 2·5482 i
10	+ 1·72454 − 0·64605 i	$-$ 7·7293 − 2·9031 i
11	+ 2·49747 − 0·35574 i	$-$ 12·7243 − 2·1916 i
12	+ 4·01964 − 0·17216 i	$-$ 27·2807 − 1·3194 i
13	+ 7·55164 − 0·07465 i	$-$ 65·5265 − 0·6764 i
14	+ 16·36978 − 0·02941 i	$-$ 170·030 − 0·3054 i
15	+ 39·92071 − 0·01062 i	$-$ 475·033 − 0·124 i
16	+ 107·3844 − 0·00353 i	$-$ 1426·33 − 0·047 i
17	+ 314·45 − 0·0010 i	$-$ 4586·2 − 0·017 i
18	+ 993·19 − 0·000 i	$-$ 15725·0 − 0·010 i
19	+ 3360·3	$-$ 57274
20	+ 12112	$-$ 220750
21	+ 46299	$-$ 897460
22	+ 186974	$-$ 38374 × 10^2

It will be seen that the imaginary part of $10^{n+1}\chi_n(10)$ tends to zero, as n increases. It is true that if we continue the calculation, having used throughout, say, 5 figures, we find that the terms begin to increase again. This, however, is but an imperfection of calculation, due to the increasing value of $\frac{1}{10}(2n+1)$ in the formula and consequent loss of accuracy, as each term is deduced from the preceding pair. Any doubt that may linger will be removed by reference to (4), according to which the imaginary term in question has the expression

$$-ir^{n+1}\left(-\frac{d}{r\,dr}\right)^n \frac{\sin r}{r}.$$

Now, if we expand $r^{-1}\sin r$ and perform the differentiations, the various terms disappear in order. For example, after the 25th operation we have

$$\left(-\frac{d}{r\,dr}\right)^{25}\frac{\sin r}{r} = \frac{50\,.\,48\,.\,.\,.\,4\,.\,2}{51\,!} - \frac{52\,.\,50\,.\,.\,.\,6\,.\,4}{53\,!}r^2 + \frac{54\,.\,.\,.\,6}{55\,!}r^4 - \&\text{c.},$$

the first term being in every case positive and the subsequent terms alternately negative and positive. The series is convergent, since the numerical values of the terms continually diminish, the ratio of consecutive terms being (when $r = 10$)

$$\frac{100}{2\,.\,53}, \qquad \frac{100}{4\,.\,55}, \qquad \frac{100}{6\,.\,57}, \qquad \&\text{c.}$$

Accordingly the first term gives a limit to the sum of the series. On introduction of the factor 10^{n+1}, this becomes

$$\frac{10^{26}}{1\,.\,3\,.\,5\,.\,.\,.\,49\,.\,51},$$

i.e., approximately $10^{-8} \times 3{\cdot}0$. *A fortiori*, when n is greater than 25, the imaginary part of $10^{n+1}\chi_n(10)$ is wholly negligible.

We can now form the coefficients of P_n under the sign of summation in (15), *i.e.*, the values of

$$i^n(2n+1)\,D_n^{-1}. \qquad \qquad (19).$$

For a reason that will presently appear, it is convenient to separate the odd and even values of n.

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n.	$i^n(2n+1)D_n^{-1}$.	n.	$i^n(2n+1)D_n^{-1}$.
0	$+0\cdot06217-0\cdot07769\,i$	1	$+0\cdot21020-0\cdot21396\,i$
2	$+0\cdot41955-0\cdot28076\,i$	3	$+0\cdot68978-0\cdot19822\,i$
4	$+0\cdot93391+0\cdot13143\,i$	5	$+0\cdot92629+0\cdot74289\,i$
6	$+0\cdot33469+1\cdot42083\,i$	7	$-0\cdot96831+1\cdot48517\,i$
8	$-2\cdot13800+0\cdot00984\,i$	9	$-0\cdot84474-2\cdot36328\,i$
10	$+2\cdot38104-0\cdot89430\,i$	11	$+0\cdot30236+1\cdot75549\,i$
12	$-0\cdot91426+0\cdot04422\,i$	13	$-0\cdot00425-0\cdot41200\,i$
14	$+0\cdot17056-0\cdot00031\,i$	15	$+0\cdot00002+0\cdot06526\,i$
16	$-0\cdot02314+0\cdot00000\,i$	17	$-0\cdot00000-0\cdot00762\,i$
18	$+0\cdot00235$	19	$+0\cdot00068\,i$
20	$-0\cdot00018$	21	$-0\cdot00005\,i$
22	$+0\cdot00001$		

In the case of $\theta = 0$, or $\mu = +1$, the P's are all equal to $+1$, and we have nothing more to do than to add together all the terms in the above table. When $\theta = 180°$, or $\mu = -1$, the even P's assume (as before) the value $+1$, but now the odd P's have a reversed sign and are equal to -1. If we add together separately the even and odd terms, and so obtain the two partial sums Σ_1 and Σ_2, then $\Sigma_1 + \Sigma_2$ will be the value of Σ for $\theta = 0$, and $\Sigma_1 - \Sigma_2$ will be the value of Σ for $\theta = 180$. And this simplification applies not merely to the special values 0 and 180, but to all intermediate pairs of angles. If $\Sigma_1 + \Sigma_2$ corresponds to θ, $\Sigma_1 - \Sigma_2$ will correspond to $180 - \theta$.

For 0 and 180 we find

$$\Sigma_1 = +1\cdot22870 + \cdot35326\,i,$$

$$\Sigma_2 = +0\cdot31135 + \cdot85436\,i;$$

whence for $\theta = 0$

$$\Sigma = 2(F + iG) = +1\cdot54005 + 1\cdot20762\,i,$$

and for $\theta = 180$

$$\Sigma = 2(F + iG) = +0\cdot91735 - 0\cdot50110\,i.$$

When $\theta = 90°$, the odd P's vanish, and the even ones have the values

$$P_0 = 1, \qquad P_2 = -\tfrac{1}{2}, \qquad P_4 = \frac{1\,.\,3}{2\,.\,4}, \qquad P_6 = -\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}, \qquad \&c.$$

For other values of θ we require tables of $P_n(\theta)$ up to about $n = 20$. That given by Professor PERRY* is limited to n less than 7, and the results are expressed only to 4 places of decimals. I have been fortunate enough to interest Professor A. LODGE in this subject, and the Appendix to this paper gives a table calculated by him containing the P's up to $n = 20$ inclusive, and for angles from 0° to 90° at intervals of 5°. As has already been suggested, the range from 0° to 90° practically covers that from 90° to 180°, inasmuch as

$$P_{2n}(90 + \theta) = P_{2n}(90 - \theta), \qquad P_{2n+1}(90 + \theta) = -P_{2n+1}(90 - \theta).$$

* 'Phil. Mag.,' vol. 32, p. 516, 1891; see also FARR, vol. 49, p. 572, 1900.

In the table of coefficients it will be observed that the highest entry occurs at $n = 10$, in accordance with an anticipation expressed in a former paper.

As will readily be understood, the multiplication by P_n and the summations involve a good deal of arithmetical labour. These operations, as well as most of the preliminary ones, have been carried out in duplicate with the assistance of Mr. C. BOUTFLOWER, of Trinity College, Cambridge.

$kc = 10.$

θ.	$2(F+iG)$.	$4(F^2+G^2)$.
0°	$+1 \cdot 54005 + 1 \cdot 20762\,i$	3·8300
5	$+1 \cdot 58407 + 1 \cdot 14959\,i$	3·8309
10	$+1 \cdot 70186 + 0 \cdot 96603\,i$	3·8295
15	$+1 \cdot 84773 + 0 \cdot 63523\,i$	3·8176
30	$+1 \cdot 52622 - 1 \cdot 17708\,i$	3·7148
45	$-1 \cdot 13754 - 1 \cdot 48453\,i$	3·4978
60	$-0 \cdot 74695 + 1 \cdot 59745\,i$	3·1098
75	$+1 \cdot 45160 - 0 \cdot 62553\,i$	2·4984
90	$-1 \cdot 31954 - 0 \cdot 09924\,i$	1·75104
105	$+0 \cdot 94204 + 0 \cdot 41681\,i$	1·06117
120	$-0 \cdot 57769 - 0 \cdot 48417\,i$	0·56815
135	$+0 \cdot 29444 + 0 \cdot 43841\,i$	0·27890
150	$-0 \cdot 08146 - 0 \cdot 35600\,i$	0·13338
165	$-0 \cdot 12081 + 0 \cdot 28341\,i$	0·09492
170	$+0 \cdot 35454 + 0 \cdot 01457\,i$	0·12591
175	$+0 \cdot 76023 - 0 \cdot 34059\,i$	0·69395
180	$+0 \cdot 91735 - 0 \cdot 50110\,i$	1·09263

The results are recorded in the annexed table and in curve B, fig. 1. The intention had been to limit the calculations to intervals of 15°, but the rapid increase in $(F^2 + G^2)$ between 165° and 180° seemed to call for the interpolation of two additional angles. This increase, corresponding to the bright point in POISSON's experiment of the shadow of a circular *disc*, is probably the most interesting feature of the results. A plot is given in fig. 1, showing the relation between the angle θ, measured from the pole, and the *intensity*, proportional to $F^2 + G^2$. It should, perhaps, be emphasised that the effect here dealt with is the intensity of the *pressure* variation, to which some percipients of sound, *e.g.*, sensitive flames, are obtuse. Thus at the antipole a sensitive flame close to the surface would not respond to a distant source, since there is at that place no periodic *motion*, as is evident from the symmetry.

I now proceed to consider the case where the source, as well as the place of observation, are situated upon the sphere; but as this is more difficult than the preceding, I shall not attempt so complete a treatment. It will be supposed still that $kc = 10$.

The analytical solution is expressed in (17), which we may compare with (15). Restricting ourselves for the present to the factors under the sign of summation, we see that the coefficient of P_n in (17) is

$$\frac{(2n+1)\,c^{n+1}\chi_n(c)}{c^{n+1}\{\chi_{n-1}(c)-(n+1)\,\chi_n(c)\}}=\frac{(2n+1)\,c^{n+1}\chi_n}{D_n},$$

while the corresponding coefficient in (15) is

$$\frac{(2n+1)\,i^n}{D_n}.$$

If these coefficients be called C_n, C'_n respectively, we have

$$C_n = i^{-n}c^{n+1}\chi_n(c) \,.\, C'_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

in which the complex factors $c^{n+1}\chi_n(c)$, C'_n, for $c=10$, have already been tabulated. We find

n.	$\frac{(2n+1)\,10^{n+1}\chi_n}{D_n}$.	n.	$\frac{(2n+1)\,10^{n+1}\chi_n}{D_n}$.
0	$-0\cdot0099+0\cdot0990\,i$	1	$-0\cdot0306+0\cdot2999\,i$
2	$-0\cdot0542+0\cdot5097\,i$	3	$-0\cdot0835+0\cdot7358\,i$
4	$-0\cdot1233+0\cdot9883\,i$	5	$-0\cdot1827+1\cdot2817\,i$
6	$-0\cdot2813+1\cdot6390\,i$	7	$-0\cdot4666+2\cdot0956\,i$
8	$-0\cdot8667+2\cdot6889\,i$	9	$-1\cdot8111+3\cdot3152\,i$
10	$-3\cdot5284+3\cdot0805\,i$	11	$-4\cdot2766+1\cdot3796\,i$
12	$-3\cdot6673+0\cdot3351\,i$	13	$-3\cdot1110+0\cdot0629\,i$
14	$-2\cdot7920+0\cdot0100\,i$	15	$-2\cdot6051+0\cdot0014\,i$
16	$-2\cdot4844+0\cdot0001\,i$	17	$-2\cdot3998$
18	$-2\cdot3369$	19	$-2\cdot2881$
20	$-2\cdot2496$	21	$-2\cdot2183$
22	$-2\cdot1925$	23	$-2\cdot1711$
24	$-2\cdot1528$	25	$-2\cdot1374$
26	$-2\cdot1240$		

The product above tabulated shows marked signs of approaching the limit -2, as n increases; so that the series (17) is divergent when $P_n=1$, *i.e.*, when $\theta=0$, as was of course to be expected. The interpretation may be followed further. By the definition of P_n, we have

$$\{1-2\alpha\,.\cos\theta+\alpha^2\}^{-\frac{1}{2}}=1+P_1\,.\,\alpha+P_2\,.\,\alpha^2+\ldots+P_n\,.\,\alpha^n+\ldots \quad . \quad (21);$$

so that, if we put $\alpha=1$,

$$1+P_1+P_2+P_3+\ldots=\frac{1}{2\sin\left(\frac{1}{2}\theta\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Thus, when θ is small, and the series tends to be divergent, we get from (17)

$$\psi = -\frac{\iint \mathrm{U}\, d\mathrm{S} \,.\, e^{ikat}}{2\pi \,.\, 2c \sin\left(\frac{1}{2}\theta\right)} \quad \ldots \ldots \ldots \quad (23);$$

and this is the correct value, seeing that $2c \sin\left(\frac{1}{2}\theta\right)$ represents the distance between the source and the point of observation, and that on account of the sphere the value of ψ is twice as great in the neighbourhood of the source as it would be were the source situated in the open.

When $\theta = 180°$, *i.e.*, at the point on the sphere immediately opposite to the source, the series converges, since P_n takes alternately the values $+1$ and -1. It will be convenient to re-tabulate continuously these values from $n = 18$ onwards without regard to sign and to exhibit the differences.

n.	Function.	First difference.	Second difference.	Third difference.
18	2·3369	—	—	—
19	2·2881	− ·0488	—	—
20	2·2496	− ·0385	+ ·0103	—
21	2·2183	− ·0313	+ ·0072	− ·0031
22	2·1925	− ·0258	+ ·0055	− ·0017
23	2·1711	− ·0214	+ ·0044	− ·0011
24	2·1528	− ·0183	+ ·0031	− ·0013
25	2·1374	− ·0154	+ ·0029	− ·0002
26	2·1240	− ·0134	+ ·0020	− ·0009

In summing the infinite series, we have to add together the terms as they actually occur up to a certain point and then estimate the value of the remainder. The simple addition is carried as far as $n = 21$ inclusive, and the result is for the even values of n

$$-18{\cdot}3939 + 9{\cdot}3506\, i,$$

and for the odd values

$$-19{\cdot}4734 + 9{\cdot}1721\, i,$$

or, with signs reversed to correspond with $\mathrm{P}_{2n+1}(180) = -1$,

$$+19{\cdot}4734 - 9{\cdot}1721\, i.$$

The complete sum up to $n = 21$ inclusive is thus

$$+1{\cdot}0795 + {\cdot}1785\, i \quad \ldots \ldots \ldots \quad (24).$$

The remainder is to be found by the methods of Finite Differences. The formula applicable to series of this kind may be written

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$$\phi(0) - \phi(1) + \phi(2) - \ldots = \tfrac{1}{2}\phi(0) - \tfrac{1}{4}\Delta\phi(0) + \tfrac{1}{8}\Delta^2\phi(0) - \ldots,$$

in which we may put

$$\phi(0) = 2{\cdot}1925, \quad \phi(1) = 2{\cdot}1711, \quad \&c.$$

Thus

$$\phi(0) - \phi(1) + \ldots = +1{\cdot}0962 + {\cdot}0054 + {\cdot}0004 = 1{\cdot}1020,$$

and for the actual remainder this is to be taken negatively. The sum of the infinite series for $\theta = 180°$ is accordingly

$$-{\cdot}0225 + {\cdot}1785\, i \quad \ldots \ldots \quad (25),$$

from which the intensity, represented by $({\cdot}0225)^2 + ({\cdot}1785)^2$, is proportional to $\cdot 03237$. Referring to (17), we see that the amplitude of ψ is in this case

$$\frac{\iint \mathrm{U}\, d\mathrm{S}}{4\pi c} \times \sqrt{({\cdot}03237)} \quad \ldots \ldots \quad (26).$$

We may compare this with the amplitude of the vibration which would occur at the same place if the sphere were removed. Here

$$\frac{\iint \mathrm{U}\, d\mathrm{S}}{4\pi r} = \frac{\iint \mathrm{U}\, d\mathrm{S}}{4\pi c} \times \sqrt{({\cdot}25)} \quad \ldots \ldots \quad (27),$$

since $r = 2c$. The effect of the sphere is therefore to reduce the intensity in the ratio of $\cdot 25$ to $\cdot 03237$.

In like manner we may treat the case of $\theta = 90°$, *i.e.*, when the point of observation is on the equator. The odd P's now vanish and the even P's take signs alternately opposite. The following table gives the values required for the direct summation, *i.e.*, up to $n = 21$ inclusive :—

n.	$\dfrac{(2n+1)\,10^{n+1}\chi_n \,.\, \mathrm{P}_n(90)}{\mathrm{D}_n}$.	n.	$\dfrac{(2n+1)\,10^{n+1}\chi_n \,.\, \mathrm{P}_n(90)}{\mathrm{D}_n}$.
0	$-\ {\cdot}0099 + {\cdot}0990\, i$	2	$+\ {\cdot}0271 - {\cdot}2548\, i$
4	$-\ {\cdot}0462 + {\cdot}3706\, i$	6	$+\ {\cdot}0879 - {\cdot}5122\, i$
8	$-\ {\cdot}2370 + {\cdot}7353\, i$	10	$+\ {\cdot}8683 - {\cdot}7581\, i$
12	$-\ {\cdot}8273 + {\cdot}0756\, i$	14	$+\ {\cdot}5848 - {\cdot}0021\, i$
16	$-\ {\cdot}4879 + {\cdot}0000\, i$	18	$+\ {\cdot}4334$
20	$-\ {\cdot}3964$		
	$-2{\cdot}0047 + 1{\cdot}2805\, i$		$+2{\cdot}0015 - 1{\cdot}5272\, i$

The next three terms, written without regard to sign, and their differences are as follows :—

22	·3688	—	—
24	·3470	− ·0218	—
26	·3292	− ·0178	+ ·0040

The remainder is found, as before, to be

$$+\tfrac{1}{2}(\cdot 3688)+\tfrac{1}{4}(\cdot 0218)+\tfrac{1}{8}(\cdot 0040)=+\cdot 1903.$$

The sum of the infinite series from the beginning is accordingly

$$+\cdot 1871-\cdot 2467\,i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

in which

$$(\cdot 1871)^2+(\cdot 2467)^2=\cdot 09588.$$

The distance between the source and the point of observation is now $2c\sin 45^\circ = c\sqrt{2}$.

The intensity in the actual case is thus ·09588 as compared with ·5 if the sphere were away.

For other angular positions than those already discussed, not only would the arithmetical work be heavier on account of the factors P_n, but the remainder would demand a more elaborated treatment.

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APPENDIX.

By Professor A. LODGE.

TABLE of Zonal Harmonics; *i.e.*, of the Coefficients of the Powers of x as far as P_{20} in the Expansion of $(1-2x\cos\theta+x^2)^{-\frac{1}{2}}$ in the form $1+P_1x+P_2x^2+\ldots+P_nx^n+\ldots$ for 5° Intervals in the Values of θ from 0° to 90°. The Table is calculated to 7 decimal places, and the last figure is approximate.

θ.	P_1 (= cos θ).	P_2.	P_3.	P_4.	P_5.
0°	1·0000000	1·0000000	1·0000000	1·0000000	1·0000000
5	·9961947	·9886059	·9772766	·9622718	·9436768
10	·9848078	·9547695	·9105688	·8532094	·7839902
15	·9659258	·8995191	·8041639	·6846954	·5471259
20	·9396926	·8245333	·6648847	·4749778	·2714918
25	·9063078	·7320907	·5016273	·2465322	+ ·0008795
30	·8660254	·6250000	·3247595	+ ·0234375	− ·2232722
35	·8191520	·5065151	+ ·1454201	− ·1714242	− ·3690967
40	·7660444	·3802362	− ·0252333	− ·3190044	− ·4196822
45	·7071068	·2500000	− ·1767767	− ·4062500	− ·3756505
50	·6427876	·1197638	− ·3002205	− ·4275344	− ·2544885
55	·5735764	− ·0065151	− ·3886125	− ·3851868	− ·0867913
60	·5000000	− ·1250000	− ·4375000	− ·2890625	+ ·0898437
65	·4226183	− ·2320907	− ·4452218	− ·1552100	·2381072
70	·3420201	− ·3245333	− ·4130083	− ·0038000	·3280672
75	·2588190	− ·3995191	− ·3448846	+ ·1434296	·3427278
80	·1736482	− ·4547695	− ·2473819	+ ·2659016	·2810175
85	·0871557	− ·4886059	− ·1290785	·3467670	·1576637
90	Nil	− ·5000000	Nil	·3750000	Nil

θ.	P_6.	P_7.	P_8.	P_9.	P_{10}.
0	1·0000000	1·0000000	1·0000000	1·0000000	1·0000000
5	·9215975	·8961595	·8675072	·8358030	·8012263
10	·7044712	·6164362	·5218462	+ ·4227908	+ ·3214371
15	·3983060	+ ·2455411	+ ·0961844	− ·0427679	− ·1650562
20	+ ·0719030	− ·1072262	− ·2518395	− ·3516966	− ·4012692
25	− ·2039822	− ·3440850	− ·4062285	− ·3895753	− ·3052371
30	− ·3740235	− ·4101780	− ·3387755	− ·1895752	− ·0070382
35	− ·4114480	− ·3095600	− ·1154393	+ ·0965467	+ ·2541595
40	− ·3235708	− ·1006016	+ ·1386270	·2900130	·2973452
45	− ·1484376	+ ·1270581	·2983398	·2855358	+ ·1151123
50	+ ·0563782	·2854345	·2946824	+ ·1040702	− ·1381136
55	·2297230	·3190966	+ ·1421667	− ·1296151	− ·2692039
60	·3232421	·2231445	− ·0736389	− ·2678985	− ·1882286
65	·3138270	+ ·0422192	− ·2411439	− ·2300283	+ ·0323225
70	·2088770	− ·1485259	− ·2780153	− ·0475854	+ ·2192910
75	+ ·0431002	− ·2730500	− ·1702200	+ ·1594939	·2316302
80	− ·1321214	− ·2834799	+ ·0233080	·2596272	+ ·0646821
85	− ·2637801	− ·1778359	·2017462	·1912893	− ·1498947
90	− ·3125000	Nil	·2734376	Nil	− ·2460938

Table of Zonal Harmonics; *i.e.*, of the Coefficients of the Powers of x as far as P_{20} in the Expansion of $(1-2x\cos\theta+x^2)^{-\frac{1}{2}}$ in the form $1+P_1x+P_2x^2+\ldots+P_nx^n+\ldots$ for 5° Intervals in the Values of θ from 0° to 90°. The Table is calculated to 7 decimal places, and the last figure is approximate—continued.

θ.	P_{11}.	P_{12}.	P_{13}.	P_{14}.	P_{15}.
0°	1·0000000	1·0000000	1·0000000	1·0000000	1·0000000
5	·7639723	·7242508	·6822849	+ ·6383094	+ ·5925694
10	+ ·2199746	+ ·1205620	+ ·0252742	− ·0639478	− ·1453436
15	− ·2654901	− ·3402156	− ·3868998	− ·4048245	− ·3948856
20	− ·4001361	− ·3528461	− ·2682722	− ·1585374	− ·0376336
25	− ·1739692	− ·0223995	+ ·1215469	+ ·2332489	+ ·2952537
30	+ ·1607048	+ ·2732027	+ ·3066580	+ ·2584895	+ ·1465789
35	·3096940	+ ·2532528	+ ·1130760	− ·0565267	− ·1950586
40	+ ·1712040	− ·0211959	− ·1892595	− ·2599246	− ·2083112
45	− ·1041843	− ·2467193	− ·2393239	− ·0972709	+ ·0903925
50	− ·2640939	− ·1987621	− ·0019170	+ ·1821884	+ ·2281988
55	− ·1769491	+ ·0522404	+ ·2209602	+ ·1959135	+ ·0110216
60	+ ·0638713	·2337529	+ ·1658041	− ·0571737	− ·2100185
65	+ ·2351950	+ ·1608831	− ·0863490	− ·2197701	− ·0989734
70	+ ·1864450	− ·0787947	− ·2239288	− ·0745390	+ ·1597121
75	− ·0305439	− ·2274796	− ·0850288	+ ·1687887	+ ·1638193
80	− ·2145820	− ·1307104	+ ·1544264	+ ·1730902	− ·0860215
85	− ·1988401	+ ·1041876	+ ·2010073	− ·0629592	− ·1982155
90	Nil	+ ·2255858	Nil	− ·2094726	Nil

θ.	P_{16}.	P_{17}.	P_{18}.	P_{19}.	P_{20}.
0	1·0000000	1·0000000	1·0000000	1·0000000	1·0000000
5	+ ·5453192	+ ·4968206	+ ·4473403	+ ·3971492	+ ·3465207
10	− ·2173739	− ·2787566	− ·3284945	− ·3658960	− ·3905880
15	− ·3594981	− ·3024136	− ·2284640	− ·1432466	− ·0527721
20	+ ·0801110	+ ·1815511	+ ·2560661	+ ·2965867	+ ·3002029
25	·2997862	+ ·2495290	+ ·1566049	+ ·0399984	− ·0780855
30	+ ·0036143	− ·1318805	− ·2254922	− ·2553464	− ·2169986
35	− ·2565851	− ·2244163	− ·1151189	+ ·0289684	+ ·1556356
40	− ·0654984	+ ·0986597	+ ·2088162	+ ·2180388	+ ·1273281
45	+ ·2150310	+ ·2100803	+ ·0857609	− ·0809310	− ·1930653
50	+ ·1133974	− ·0732822	− ·1986904	− ·1792842	− ·0359655
55	− ·1714205	− ·2012353	− ·0625381	+ ·1207910	+ ·1945128
60	− ·1498551	+ ·0522168	+ ·1922962	+ ·1377671	− ·0483584
65	+ ·1249926	+ ·1956924	+ ·0427633	− ·1501989	− ·1644051
70	+ ·1757158	− ·0336558	− ·1883363	− ·0935549	+ ·1165241
75	− ·0760903	− ·1924117	− ·0249700	+ ·1696995	+ ·1093683
80	− ·1912133	+ ·0165069	+ ·1861639	+ ·0473145	− ·1608344
85	+ ·0255526	+ ·1908789	+ ·0082151	− ·1794383	− ·0383005
90	+ ·1963808	Nil	− ·1854706	Nil	+ ·1761970

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Explanation of the Methods of Compiling and Checking the above Tables.

Calculation of the Even Orders.

The Zonal Harmonics of even order in the foregoing tables were calculated from the formula obtained by expanding $(1 - 2x \cos \theta + x^2)^{-\frac{1}{2}}$ by means of the form

$$(1 - xe^{i\theta})^{-\frac{1}{2}} (1 - xe^{-i\theta})^{-\frac{1}{2}},$$

i.e.,

$$(a_0 + a_1 xe^{i\theta} + \ldots + a_n x^n e^{ni\theta} + \ldots + a_{2n} x^{2n} e^{2ni\theta} + \ldots)$$
$$\times (a_0 + a_1 xe^{-i\theta} + \ldots + a_n x^n e^{-ni\theta} + \ldots + a_{2n} x^{2n} e^{-2ni\theta} + \ldots),$$

where

$$a_r = \frac{1 . 3 . 5 \ldots (2r - 1)}{2 . 4 . 6 \ldots 2r}, \quad \text{and} \quad a_0 = 1;$$

whence

$$P_{2n}(\theta) = a_n^2 + 2a_{n-1}a_{n+1} \cos 2\theta + \ldots + 2a_0 a_{2n} \cos 2n\theta.$$

To calculate the coefficients a_n^2, $2a_{n-1}a_{n+1}$, . . . $2a_0 a_{2n}$, an auxiliary table of values of $\log_{10} a_r$ was formed from $r = 1$ to $r = 20$, to 8 decimal places; and a similar table of $\log_{10} 2a_r$ from $r = 0$ to $r = 9$; so as readily to combine them to form (to 7 decimal places) the logarithms of the required coefficients for different values of n.

The coefficients were then calculated to 7 decimal places from their logarithms, and checked for each value of n by seeing that they added up to unity in each case.

Next, a table of values of $\log \cos 2\theta$, $\log \cos 4\theta$, . . . to 7 decimals, was formed for all values of θ, at 5° intervals, from 5° to 90°. The addition of these to the logarithms of the corresponding coefficients gave the logarithms of the various terms (except as regards sign) in the above expansion of $P_{2n}(\theta)$. From these logarithms the terms themselves were calculated to 7 decimals and tabulated, the positive terms in black, and the negative terms in red ink. The accuracy of these terms was checked by making use of the identities

(1.) $2 \cos 60° = 1$,

(2.) $\cos 50° + \cos 70° = \cos 10°$,

(3.) $\cos 40° + \cos 80° = \cos 20°$.

This, in addition to the primary identity

$$a_n^2 + 2a_{n-1}a_{n+1} + \ldots + 2a_0 a_{2n} = 1,$$

checked all the terms effectually except those which were multiples of $\cos 30°$. These were checked by adding a number of them together and comparing their sum with the sum of the coefficients multiplied in a lump by $\cos 30°$.

In these ways all the separate terms were ensured to be free from errors due to carelessness in taking proportional parts, or any other incidental errors.

Then the terms were added together for each value of θ in $P_{2n}(\theta)$ for a given value of n, so obtaining the values required for the actual table. By *adding* I mean to include also subtracting, the artifice of putting positive terms in black and negative terms in red being a great help in this part of the work. Errors in this work were corrected by adding all the values of $P_{2n}(\theta)$ from $\theta = 5°$ to $\theta = 90°$ for a given value of n, and comparing the result with the sum obtained in a different way (see note at the end of the second auxiliary table appended). Up to P_{12} the additions and subtractions and checkings were all done without mechanical aid, but for the later values of n, from P_{14} to P_{20}, I made use of an EDMONDSON'S calculating machine which was very kindly lent to me by Professor McLEOD.

In this way all the even harmonics were calculated and were ensured to be free from errors, except those incidental to the last figure, which is, of course, only approximate, as the terms used in the calculation were evaluated to 7 decimals only. I am confident, however, that the last figure is never far from the real value, and that it would be more accurate in every case to retain it in numerical work with the tables than to omit it. The error is not usually more than ± 2 in the 7th place, and I am confident that it never exceeds ± 3, whereas omitting it would lead to a possibility of ± 5 in *addition* to its actual error, *i.e.*, to a maximum error of ± 8. I have assumed that there are very few numerical calculations requiring an accuracy greater than an approximate 7th decimal place, and that, therefore, the vastly increased difficulty which would have been caused by working throughout with 8 decimals would have been wasted labour.

Calculation of the Odd Orders, and Final Checking.

When the even orders were calculated, the question arose as to the best way of calculating the odd orders. P_1, of course, gave no difficulty, being merely $\cos\theta$. P_3, also, was quite easy to calculate directly from its value $\frac{1}{8}(3\cos\theta + 5\cos 3\theta)$, EDMONDSON'S machine being used for the purpose. The remaining odd functions were calculated from the even ones by means of the identity

$$(2n-1)\,P_1P_{n-1} = nP_n + (n-1)\,P_{n-2}.$$

The accuracy of the results was checked by recalculating the even P's from the odd ones by means of the same formula. This clinched everything.

The mode of using this formula which I adopted, between $\theta = 5°$ and $\theta = 60°$ inclusive, was different from that adopted between $\theta = 65°$ and $\theta = 85°$ inclusive, so as to minimize the effect of 7th-figure inaccuracies as much as possible.

Up to $\theta = 60°$ I used it in the form

$$P_n = \frac{(n+1) P_{n+1} + n P_{n-1}}{(2n+1) P_1},$$

where P_1 varied from 1 to $\frac{1}{2}$; each P being thus dependent on its immediate predecessor and successor.

Beyond 60° I thought it better to use it in the progressive form

$$P_n = \frac{(2n-1) P_1 P_{n-1} - (n-1) P_{n-2}}{n},$$

each P being thus calculated from the two preceding orders.

I believe that in this way the maximum risk of a 7th-figure error occurs at 60°, when $P_1 = \frac{1}{2}$, and is not very great even there, whereas the exclusive use of either method would have greatly magnified the error at one end or other of the table.

Auxiliary Tables.

TABLE of Values of log a_r and log $2a_r$.

r.	log a_r.	log $2a_r$.
0	Nil	0·30103000
1	$\bar{1}$·69897000	Nil
2	$\bar{1}$·57403127	$\bar{1}$·87506126
3	$\bar{1}$·49485002	$\bar{1}$·79588002
4	$\bar{1}$·43685807	$\bar{1}$·73788807
5	$\bar{1}$·39110058	$\bar{1}$·69213058
6	$\bar{1}$·35331202	$\bar{1}$·65434202
7	$\bar{1}$·32112734	$\bar{1}$·62215734
8	$\bar{1}$·29309862	$\bar{1}$·59412861
9	$\bar{1}$·26827503	$\bar{1}$·56930503
10	$\bar{1}$·24599864	
11	$\bar{1}$·22579525	
12	$\bar{1}$·20731185	
13	$\bar{1}$·19027851	
14	$\bar{1}$·17448424	
15	$\bar{1}$·15976098	
16	$\bar{1}$·14597270	
17	$\bar{1}$·13300772	
18	$\bar{1}$·12077326	
19	$\bar{1}$·10919139	
20	$\bar{1}$·09819601	

TABLE showing the Acute Angles (in degrees) whose Cosines were required in Forming the Terms belonging to the Harmonics of Even Order. The Signs prefixed to the Angles Indicate whether their Cosines had to be Added or Subtracted.

θ.	2θ.	4θ.	6θ.	8θ.	10θ.	12θ.	14θ.	16θ.	18θ.	20θ.
5	+10	+20	+30	+40	+50	+60	+70	+80	±90	−80
10	+20	+40	+60	+80	−80	−60	−40	−20	− 0	−20
15	+30	+60	±90	−60	−30	− 0	−30	−60	∓90	+60
20	+40	+80	−60	−20	−20	−60	+80	+40	+ 0	+40
25	+50	−80	−30	−20	−70	+60	+10	+40	±90	−40
30	+60	−60	− 0	−60	+60	+ 0	+60	−60	− 0	−60
35	+70	−40	−30	+80	+10	+60	−50	−20	∓90	+20
40	+80	−20	−60	+40	+40	−60	−20	+80	+ 0	+80
45	±90	− 0	∓90	+ 0	±90	− 0	∓90	+ 0	±90	− 0
50	−80	−20	+60	+40	−40	−60	+20	+80	− 0	+80
55	−70	−40	+30	+80	−10	+60	+50	−20	∓90	+20
60	−60	−60	+ 0	−60	−60	+ 0	−60	−60	+ 0	−60
65	−50	−80	+30	−20	+70	+60	−10	+40	±90	−40
70	−40	+80	+60	−20	+20	−60	−80	+40	− 0	+40
75	−30	+60	±90	−60	+30	− 0	+30	−60	∓90	+60
80	−20	+40	−60	+80	+80	−60	+40	−20	+ 0	−20
85	−10	+20	−30	+40	−50	+60	−70	+80	±90	−80
90	− 0	+ 0	− 0	+ 0	− 0	+ 0	− 0	+ 0	− 0	+ 0

Note.—The terms in each of the columns headed 4θ, 8θ, 12θ, . . . all balance, their sum being zero.

The terms in each of the columns headed 2θ, 6θ, 10θ, . . . balance except the last term.

Hence the sum of all the 18 values of $P_n(\theta)$, from 5° to 90°,

$$= 18a_n^2 - 2a_{n-1}a_{n+1} - 2a_{n-3}a_{n+3} - \ldots$$

This equivalence was made use of in finally checking the values of the harmonics of each even order.

Some Characteristics of the Functions as shown by the Tables.

The functions become more and more undulating as n increases, $P_n(\theta)$ having n zeroes between $\theta = 0°$ and 180°, similarly spaced on either side of 90°. The most remarkable peculiarity noticeable in drawing their graphs is that the intervals between the successive zeroes from the first to the n^{th} are almost exactly equal.

The graph of P_{20} is reproduced in fig. 2, to emphasize this peculiarity of equal intervals.

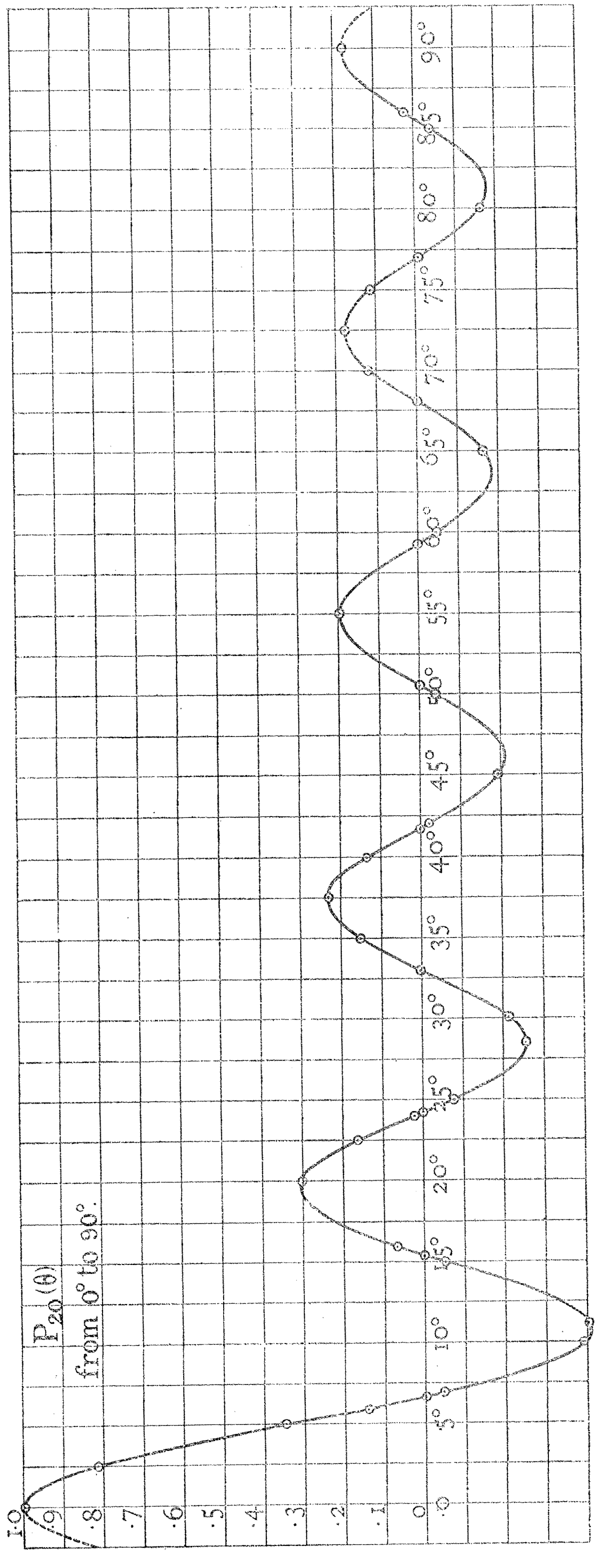

Fig. 2.

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In this respect LAPLACE'S approximate formula for high values of n, viz. :—

$$P_n(\theta) = \frac{\sqrt{2}}{\sqrt{(n\pi \sin\theta)}} \cos\left(n\theta + \frac{\theta}{2} - \frac{\pi}{4}\right),$$

shows a wonderful resemblance to the actual functions even for quite low values of n.

The *numerical* values of this function are, indeed, not very near the true values even when $n = 20$, as will be seen by the following short table :—

	Approximation.	True value.
$P_{20}(15°)$	$-\cdot 04577$	$-\cdot 05277$
30°	$-\cdot 21851$	$-\cdot 21700$
45°	$-\cdot 19602$	$-\cdot 19307$
60°	$-\cdot 04962$	$-\cdot 04836$
75°	$+\cdot 11051$	$+\cdot 10937$
90°	$+\cdot 17841$	$+\cdot 17620$

But, though its numerical values are not very close, the positions of most of its zeroes are remarkably near the correct places. It can, of course, only be considered between 0° and 180°, since $\sin\theta$ becomes negative beyond these limits. But between these limits it has n zeroes, with $n - 1$ equal intervals between them, the first zero being at $\theta = 270° \div (2n + 1)$, and the interval between successive zeroes being $360° \div (2n + 1)$, the formula for the required values of θ being $\theta = (4r + 3)\, 90° \div (2n + 1)$, for integer values of r from 0 to $n - 1$.

Taking $n = 20$, this would make the first zero approximately at 6° 35′, and the constant interval 8° 47′, very nearly; the roots given by the formula being, roughly,

$$6° \, 35', \; 15° \, 22', \; 24° \, 9', \; 32° \, 55\tfrac{1}{2}', \; 41° \, 42\tfrac{1}{2}', \; 50° \, 29', \; 59° \, 16', \ldots$$

The actual value of the first root of $P_{20}(\theta) = 0$ is slightly over 6° 43′, and intervals between successive roots are very nearly equal, varying between 8° 43′ and 8° 47′.

The first ten roots are, to something like the nearest minute, 6° 43′, 15° 26′, 24° 11′, 32° 57′, 41° 44′, 50° 30′, 59° 16′, 68° 3′, 76° 50′, and 85° 37′.

Professor PERRY has brought out a table of Zonal Harmonics to 4 decimals, for every degree, as far as P_7 (' Phil. Mag.,' December, 1891), and by help of this table I have calculated the first root, and the intervals between successive roots, for P_3 to P_7, to something like 1 minute accuracy. Their values, and the corresponding approximations obtained from LAPLACE'S formula above, are given in the following table, showing how far they differ for these low values of n :—

	First root, and successive intervals.	LAPLACE'S approximation.
P_3	39° 14′	38° 34′
	50 46	51 26
	50 46	
P_4	30 33	30 0
	39 34½	40 0
	39 45	
	39 34½	
P_5	25 1	24 33
	32 24	32 49
	32 35	
	32 35	
	32 24	
P_6	21° 11′	20° 46′
	27 26	27 41½
	27 35	
	27 36	
	27 35	
	27 26	
P_7	18 24	18 0
	23 45	24 0
	23 54	
	23 57	
	23 57	
	23 54	
	23 45	

These examples indicate that the first root is always greater than the value given by the approximate formula, and the successive intervals are slightly less.

The actual roots of P_7 are, approximately,

$$18^\circ\ 24', \quad 42^\circ\ 9', \quad 66^\circ\ 3', \quad 90^\circ, \text{ \&c.},$$

and those given by the Laplace formula are

$$18^\circ, \quad 42^\circ, \quad 66^\circ, \quad 90^\circ,$$

so that the true roots are all a little ahead of those given by the Laplace formula, so long as θ is less than 90°. Beyond 90° the roots are, of course, similarly spaced in reverse order.

NOTE BY LORD RAYLEIGH.

Professor LODGE'S comparison of P_{20} with LAPLACE'S approximate value suggests the question whether it is possible to effect an improvement in the approximate expression without entailing too great a complication. The following, on the lines of the investigation in TODHUNTER'S 'Functions of LAPLACE, &c.,'* § 89, shows, I think, that this can be done.

* MACMILLAN and Co., London, 1875.

We have

$$P_n = \frac{4}{\pi k(2n+1)}\left\{\sin(n+1)\theta\right.$$

$$\left. + \frac{1.(n+1)}{1.(2n+3)}\sin(n+3)\theta + \frac{1.3.(n+1)(n+2)}{1.2.(2n+3)(2n+5)}\sin(n+5)\theta + \ldots\right\} \quad . \quad . \quad (\alpha),$$

with

$$\frac{1}{k} = \frac{2.4.6\ldots 2n}{1.3.5\ldots(2n-1)} = \sqrt{(\pi n)}.\left\{1 + \frac{1}{8n} + \ldots\right\} \quad . \quad . \quad . \quad . \quad (\beta).$$

When n is great, approximate values may be used for the coefficients of the sines in (α). To obtain LAPLACE's expression it suffices to take

$$1, \quad \frac{1}{2}, \quad \frac{1.3}{2.4}, \quad \frac{1.3.5}{2.4.6}, \quad \text{\&c.};$$

but now we require a closer approximation. Thus

$$\frac{1.(n+1)}{1.(2n+3)} = \frac{1}{2}\left(1 - \frac{1}{2n+2}\right),$$

$$\frac{1.3.(n+1).(n+2)}{1.2.(2n+3).(2n+5)} = \frac{1.3}{2.4}\left(1 - \frac{1}{2n+2} - \frac{1}{2n+4}\right),$$

and so on. If we write

$$x = 1 - \frac{1}{2n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma),$$

the coefficients are approximately

$$1, \quad \frac{1}{2}x, \quad \frac{1.3}{2.4}x^2, \quad \frac{1.3.5}{2.4.6}x^3, \quad \text{\&c.},$$

and the series takes actually the form assumed by TODHUNTER for analytical convenience. In his notation

$$C = t\cos\theta + \tfrac{1}{2}t^3\cos 3\theta + \frac{1.3}{2.4}t^5\cos 5\theta + \ldots,$$

$$S = t\sin\theta + \tfrac{1}{2}t^3\sin 3\theta + \frac{1.3}{2.4}t^5\sin 5\theta + \ldots,$$

and

$$P_n = \frac{4}{\pi k(2n+1)}\{C\sin n\theta + S\cos n\theta\},$$

where ultimately t is to be made equal to unity.

By summation of the series ($t < 1$),

$$C = \frac{t}{\sqrt{\rho}}\cos(\theta + \tfrac{1}{2}\phi), \quad S = \frac{t}{\sqrt{\rho}}\sin(\theta + \tfrac{1}{2}\phi),$$

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where

$$\rho^2 = 1 - 2t^2 \cos 2\theta + t^4, \qquad \tan\phi = \frac{t^2 \sin 2\theta}{1 - t^2 \cos 2\theta} \quad . \quad . \quad . \quad . \quad . \quad (\delta).$$

For our purpose it is only necessary to write C/t and S/t for C and S respectively, and to identify t^2 with x in (γ). Thus

$$\mathrm{P}_n = \frac{4}{\pi k (2n+1)\sqrt{\rho}} \sin\left(n\theta + \theta + \tfrac{1}{2}\phi\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\epsilon),$$

ρ and ϕ being given by (δ). We find, with $t = 1 - \dfrac{1}{4n}$,

$$\rho^2 = 4\sin^2\theta\left(1 - \frac{1}{2n}\right),$$

so that

$$\sqrt{\rho} = \sqrt{(2\sin\theta)} \cdot \left(1 - \frac{1}{8n}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\zeta);$$

and

$$\tan\phi = \frac{\sin 2\theta}{2\sin^2\theta + 1/2n},$$

whence

$$\phi = \frac{\pi}{2} - \theta - \frac{\cot\theta}{4n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\eta).$$

Using (ζ), (η), (β) in (ϵ) we get

$$\mathrm{P}_n = \frac{\sqrt{2}}{\sqrt{(\pi n \sin\theta)}}\left\{1 - \frac{1}{4n}\right\} . \cos\left\{n\theta + \frac{\theta}{2} - \frac{\pi}{4} - \frac{\cot\theta}{8n}\right\} \quad . \quad . \quad . \quad (\theta),$$

which is the expression required.

By this extension, not only is a closer approximation obtained, but the logic of the process is improved.

A comparison of values according to (θ) with the true values may be given in the case of n equal to 20.

VALUES of P_{20}.

θ.	True value.	According to (θ).
15°	−·05277	−·05320
30	−·21700	−·21712
45	−·19307	−·19306
60	−·04836	−·04834
75	+·10937	+·10937
90	+·17620	+·17618